Numerical Method for Solving Stiff System of Ordinary Differential Equations by using Differential Transform Method

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Abstract

In this paper, the differential transform method is applied to build the exact solution for stiff systems of ordinary differential equations. It has been watched that the proposed method is very efficient and accountable for the solution of stiff systems of ordinary differential equations. Numerical outcomes illustrate the achievement and strength of the proposed method. The differential transformation method (DTM)is utilized to calculate an approximation to the solution of the stiff systems of ordinary differential equations. The outcomes are compared with the results obtained by different numerical methods. Some applications are approaching to show the precision and simplicity of the method.

Keywords: Differential Transform, Series solution, Stiff systems of ordinary differential equations.

الحل العددي لنظام صلب من المعادلات التفاضلية الاعتيادية باستخدام طريقة تحويل المشتقة

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المستخلص

في هذا البحث, تم تطبيق طريقة تحويل المشتقة لبناء الحل الدقيق للأنظمة الصلبة للمعادلات التفاضلية الاعتيادية.لقد لوحظ أن الطريقة المقترحة فعالة للغاية ومسؤولة عن حل الأنظمة الصلبة للمعادلات التفاضلية الاعتيادية. توضح النتائج العددية الإنجاز وقوة الطريقة المقترحة. تم استخدام طريقة تحويل المشتقة (DTM) لحساب تقريب حل الأنظمة الصلبة للمعادلات التفاضلية الاعتيادية. تمت مقارنة النتائج مع النتائج التي تم الحصول عليها بطرق عددية مختلفة. تقترب بعض التطبيقات لاظهار دقة وبساطة الطريقة.

الكلمات المفتاحية: تحويل المشتقة، حل المتسلسلة، نظام صلب من المعادلات التفاضلية الاعتيادية.

Introductions

The connotation of Differential Transform Method (DTM) was foremost suggested by Zhou and proves that DTM is an iterative procedure for getting analytic Taylor's series solution of differential equations. DTM is highly advantageous to solve equation in ordinary differential equation. It is as well used to solve boundary value problem [10]**.** Abdallah and Shama used Differential Transform Method to solve Systems of ordinary Differential [2]. Abdel-Halim used Differential Transform Technique for solving higher order initial value problems [3].

The mathematical equations modelling many real-world physical phenomena are often stiff equations, i.e., equations with a wide range of temporal scales. The numerical methods for solving stiff equations must have good accuracy and wide region of stability [6]. Hippolyte, Eugene and Renovat applied the Explicit Fatuma's Method for solving First-Order Stiff Systems of Scalar Ordinary Differential Equations [4]. M. Y. Liu, L. Zhang, and C. F. Zhang, Study of Banded Implicit Runge–Kutta Methods for Solving Stiff Differential Equations [5]. Sabo,Pius and A.I.Bakari used the Modification of Second Derivatie Linear Multistep Ordinary differential Equation for solving stiffly Differential Equation[7]. Y. Skwame, J.Sunday, T.Y.Kyagya used an A-Stable Backward Difference Second Order Linear Multistep Method for solving stiff ordinary Differential Equation [8].

This paper is organized as follows. The definition Differential transform method are given in section two. Applications is given in section three. Abbrief of the conclusion is in the final section.

Differential transform method (DTM)

The transformation of the mth derivative of a function in on variable is a follow:

$$Z(m) = \frac{1}{m!}[\frac{d^m z}{dt^m}(t)]_{t=t_0} \qquad (1)$$

and the inverse transformation is known by,

$$z(t) = \sum_{m=0}^{\infty} Z(m)(t-t_0)^m \quad (2)$$

Look at[11]

The next theorems that can be extract from equations (1) and (2) are specific down:

Theorem1. If $z(t) = z_1(t) \pm z_2(t)$, then $Z_1(m) \pm Z_2(m)$.

Theorem2. If $z(t) = az_1(t)$, then $Z(m) = aZ_1(m)$, where a is a constant.

Theorem3. If $z(t) = \frac{d^n z_1(t)}{dt^n}$, then $Z(m) = \frac{(m+n)!}{m!} Z_1(m+n)$.

Theorem4. If $z(t) = z_1(t)z_2(t)$,

then $Z(m) = \sum_{m_1}^{m} Z_1(m_1)Z_2(m-m_1)$.

Theorem5. If $z(t) = t^n$, then $Z(m) = \beta(m-n)$ where $\beta(m-n) = \begin{cases} 1 & m = n \\ 0 & m \neq n \end{cases}$.

Theorem6. If $z(t) = z_1(t)z_2(t) \dots z_n(t)$, then

$$Z(m) = \sum_{m_{n-1}=0}^{m} \sum_{m_{n-2}=0}^{m_{n-1}} \dots \sum_{m_2=0}^{m_3} \sum_{m_1=0}^{m_2} Z_1(m_1)Z_2(m_2 - m_1) \dots Z_{n-1}(m_{n-1} - m_{n-2})Z_n(m - m_{n-1}).$$

Theorem7. If $z(t) = e^{\alpha t}$, then Z $(m) = \frac{\alpha^m}{m!}$, where α is a constant.

Theorem8. If $z(t) = \sin(\mu t + \rho)$, then

$Z(m) = \frac{\mu^m}{m!}\sin\left(\frac{m\pi}{2} + \rho\right)$, where μ and ρ is constant.

Theorem9. If $z(t) = \cos(\mu t + \rho)$, then

$Z(m) = \frac{\mu^m}{m!}\cos\left(\frac{m\pi}{2} + \rho\right)$, when μ and ρ constants.

The proofs of theorems 1-6 are obtainable in [1] and the proofs of Theorems 7-9 are obtainable in [9].

Applications

In this part, we apply the proposed differential transform method to stiff systems of ordinary differential equations.

Example 3.1 Consider the stiffly problem [7].

$$\frac{dz_1}{dt} - 198z_1(t) - 199z_2(t) = 0 \qquad (5)$$

$$\frac{dz_2}{dt} + 398z_1(t) + 399z_2(t) = 0 \qquad (6)$$

With the initial condition

$$z_1(0) = 1, z_2(0) = -1 \qquad (7)$$

The exact solution of this problem is:

$$z_1(t) = \mathrm{e}^{-t}, z_2(t) = -\mathrm{e}^{-t}, x \in [0,1] \qquad (8)$$

using differential transform method to the equation (5) and (6), utilizing up aforesaid we get

$$(m+1)Z_1(m+1) - 198Z_1(m) - 199Z_2(m) = 0 \qquad (9)$$

$$(m+1)Z_2(m+1) + 398Z_1(m) + 399Z_2(m) = 0 \qquad (10)$$

With initial condition $z_1(0) = 1, z_2(0) = -1$

when m=0→ $Z_1(1) = -1, Z_2(1) = 1$

when m=1→ $Z_1(2) = \frac{1}{2}, Z_2(2) = \frac{-1}{2}$

when m=2→ $Z_1(3) = \frac{-1}{6}, Z_2(3) = \frac{1}{6}$

when m=3→ $Z_1(4) = \frac{1}{24}, Z_2(4) = \frac{-1}{24}$

when m=4→ $Z_1(5) = \frac{-1}{120}, Z_2(5) = \frac{1}{120}$

when m=5→ $Z_1(6) = \frac{1}{720}, Z_2(6) = \frac{-1}{720}$

when m=6→ $Z_1(7) = \frac{-1}{5040}, Z_2(7) = \frac{1}{5040}$

when m=7→ $Z_1(8) = \frac{1}{40320}, Z_2(8) = \frac{-1}{40320}$

thus, the solution when n=8
$z_i(t) = \sum_{m=0}^{8} t^m Z_i(m)$ for i=1,2

$$z_1(t) = 1 - t + \frac{1}{2!}t^2 - \frac{1}{3!}t^3 + \frac{1}{4!}t^4 - \frac{1}{5!}t^5 + \frac{1}{6!}t^6 - \frac{1}{7!}t^7 + \frac{1}{8!}t^8$$

$$z_2(t) = -1 + t - \frac{1}{2!}t^2 + \frac{1}{3!}t^3 - \frac{1}{4!}t^4 + \frac{1}{5!}t^5 - \frac{1}{6!}t^6 + \frac{1}{7!}t^7 - \frac{1}{8!}t^8$$

Table (1)
The results of Example (1) (N=4)

t	N=4					
	Exact $z_1(t)$	Exact $z_2(t)$	Approximate $z_1(t)$	Approximate $z_2(t)$	Error of $z_1(t)$	Error of $z_2(t)$
0	1.0000	-1.0000	1.0000	-1.0000	0.0000	0.0000
0.1	0.9048	-0.9048	0.9048	-0.9048	0.0000	0.0000
0.2	0.8187	-0.8187	0.8187	-0.8187	0.0000	0.0000
0.3	0.7408	-0.7408	0.7408	-0.7408	0.0000	0.0000
0.4	0.6703	-0.6703	0.6704	-0.6704	0.0001	0.0001
0.5	0.6065	-0.6065	0.6068	-0.6068	0.0003	0.0003
0.6	0.5488	-0.5488	0.5494	-0.5494	0.0006	0.0006
0.7	0.4966	-0.4966	0.4978	-0.4978	0.0012	0.0012
0.8	0.4493	-0.4493	0.4517	-0.4517	0.0024	0.0024
0.9	0.4066	-0.4066	0.4108	-0.4108	0.0042	0.0042
1	0.3679	-0.3679	0.3750	-0.3750	0.0071	0.0071

Table (2)
The results of Example (1) (N=6)

t	N=6					
	Exact $z_1(t)$	Exact $z_2(t)$	Approximate $z_1(t)$	Approximate $z_2(t)$	Error of $z_1(t)$	Error of $z_2(t)$
0	1.0000	-1.0000	1.0000	-1.0000	0.0000	0.0000
0.1	0.9048	-0.9048	0.9048	-0.9048	0.0000	0.0000
0.2	0.8187	-0.8187	0.8187	-0.8187	0.0000	0.0000
0.3	0.7408	-0.7408	0.7408	-0.7408	0.0000	0.0000
0.4	0.6703	-0.6703	0.6703	-0.6703	0.0000	0.0000
0.5	0.6065	-0.6065	0.6065	-0.6065	0.0000	0.0000
0.6	0.5488	-0.5488	0.5488	-0.5488	0.0000	0.0000
0.7	0.4966	-0.4966	0.4966	-0.4966	0.0000	0.0000
0.8	0.4493	-0.4493	0.4494	-0.4494	0.0001	0.0001
0.9	0.4066	-0.4066	0.4067	-0.4067	0.0001	0.0001
1	0.3679	-0.3679	0.3681	-0.3681	0.0002	0.0002

Table (3)
The results of Example (1) (N=8)

t	N=8					
	Exact $z_1(t)$	Exact $z_2(t)$	Approximate $z_1(t)$	Approximate $z_2(t)$	Error of $z_1(t)$	Error of $z_2(t)$
0	1.0000	-1.0000	1.0000	-1.0000	0.0000	0.0000
0.1	0.9048	-0.9048	0.9048	-0.9048	0.0000	0.0000
0.2	0.8187	-0.8187	0.8187	-0.8187	0.0000	0.0000
0.3	0.7408	-0.7408	0.7408	-0.7408	0.0000	0.0000
0.4	0.6703	-0.6703	0.6703	-0.6703	0.0000	0.0000
0.5	0.6065	-0.6065	0.6065	-0.6065	0.0000	0.0000
0.6	0.5488	-0.5488	0.5488	-0.5488	0.0000	0.0000
0.7	0.4966	-0.4966	0.4966	-0.4966	0.0000	0.0000
0.8	0.4493	-0.4493	0.4493	-0.4493	0.0000	0.0000
0.9	0.4066	-0.4066	0.4066	-0.4066	0.0000	0.0000
1	0.3679	-0.3679	0.3679	-0.3679	0.0000	0.0000

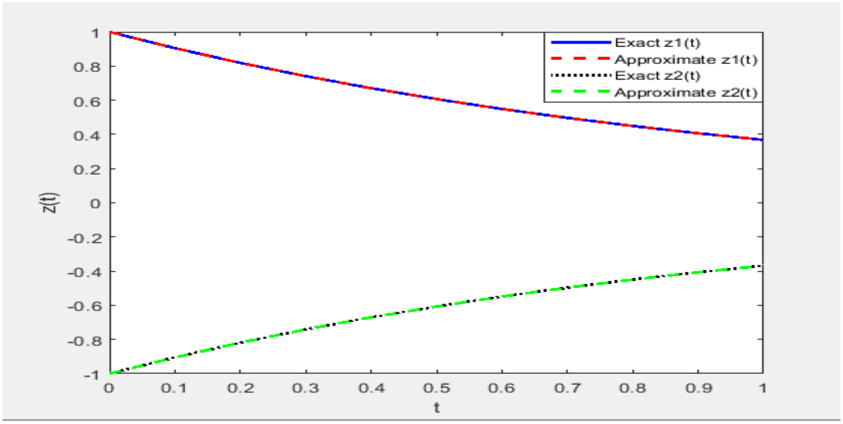

Figure (1)
Exact and Approximate solution of Example (1) (when N=8)

Example 3.2 Consider the stiffly problem [7]

$$\frac{dz_1}{dt} + 100z_1(t) - 9.901z_2(t) = 0 \qquad (11)$$

$$\frac{dz_2}{dt} - 0.1z_1(t) + z_2(t) = 0 \qquad (12)$$

with the initial condition

$$z_1(0) = 1, z_2(0) = 10 \qquad (13)$$

The exact solution of this problem is $z_1(t) = e^{-0.99t}, z_2(t) = 10e^{-0.99t}$ $x \in [0,1]$ (14)

Taking the differential transform method to the equation (11) and (12), utilizing up aforesaid we get

$$(m+1)Z_1(m+1) + 100Z_1(m) - 9.901Z_2(m) = 0$$

$$(m+1)Z_2(m+1) - 0.1Z_1(m) + Z_2(m) = 0$$

with the initial condition$z_1(0) = 1, z_2(0) = 10$

when m=0→ $Z_1(1) = -0.99, Z_2(1) = -9.9$

when m=1→ $Z_1(2) = 0.49005, Z_2(2) = 4.9005$

when m=2→ $Z_1(3) = -0.1617165, Z_2(3) = -1.617165$

when m=3→ $Z_1(4) = 0.040024832, Z_2(4) = 0.400248337$

when m=4→ $Z_1(5) = -0.007924883, Z_2(5) = -0.07924917$

when m=5→ $Z_1(6) = 0.001307044, Z_2(6) = 0.013076113$

when m=6→ $Z_1(7) = -0.000176829, Z_2(7) = -0.001849344$

when m=7→ $Z_1(8) = -0.000078431, Z_2(8) = 0.000228957$

thus, the solution when n=8

$z_i(t) = \sum_{m=0}^{8} t^m Z_i(m)$ for i=1,2'

$$z_1(t) = 1 - 0.99t + 0.49005t^2 - 0.1617165t^3 + 0.040024832t^4 - 0.007924883t^5 + 0.001307044t^6 - 0.000176829t^7 - 0.000078431t^8$$

$$z_2(t) = 10 - 9.9t + 4.9005t^2 - 1.617165t^3 + 0.400248337t^4 - 0.07924917t^5 + 0.013076113t^6 - 0.001849344t^7 + 0.000228957t^8$$

Table (4)

The results of Example (2) (N=4)

t	N=4					
	Exact $z_1(t)$	Exact $z_2(t)$	Approximate $z_1(t)$	Approximate $z_2(t)$	Error of $z_1(t)$	Error of $z_2(t)$
0	1.0000	10.0000	1.0000	10.0000	0.0000	0.0000
0.1	0.9057	9.0574	0.9057	9.0574	0.0000	0.0000
0.2	0.8204	8.2037	0.8204	8.2037	0.0000	0.0000
0.3	0.7430	7.4304	0.7431	7.4306	0.0001	0.0002
0.4	0.6730	6.7301	0.6731	6.7308	0.0001	0.0007
0.5	0.6096	6.0957	0.6098	6.0980	0.0002	0.0023
0.6	0.5521	5.5211	0.5527	5.5267	0.0006	0.0056
0.7	0.5001	5.0007	0.5013	5.0127	0.0012	0.012
0.8	0.4529	4.5294	0.4552	4.5523	0.0023	0.0229
0.9	0.4102	4.1025	0.4143	4.1431	0.0041	0.0406
1	0.3716	3.7158	0.3784	3.7836	0.0068	0.0678

Table (5)
The results of Example (2) (N=6)

t	N=6					
	Exact $z_1(t)$	Exact $z_2(t)$	Approximate $z_1(t)$	Approximate $z_2(t)$	Error of $z_1(t)$	Error of $z_2(t)$
0	1.0000	10.0000	1.0000	10.0000	0.0000	0.0000
0.1	0.9057	9.0574	0.9057	9.0574	0.0000	0.0000
0.2	0.8204	8.2037	0.8204	8.2037	0.0000	0.0000
0.3	0.7430	7.4304	0.7430	7.4304	0.0000	0.0000
0.4	0.6730	6.7301	0.6730	6.7301	0.0000	0.0000
0.5	0.6096	6.0957	0.6096	6.0957	0.0000	0.0000
0.6	0.5521	5.5211	0.5521	5.5212	0.0000	0.0001
0.7	0.5001	5.0007	0.5001	5.0009	0.0000	0.0002
0.8	0.4529	4.5294	0.4530	4.5297	0.0001	0.0003
0.9	0.4102	4.1025	0.4103	4.1032	0.0001	0.0007
1	0.3716	3.7158	0.3717	3.7174	0.0001	0.0016

Table (6)
The results of Example (2) (N=8)

t	N=8					
	Exact z_1 (t)	Exact z_2 (t)	Approximate $z_1(t)$	Approximate $z_2(t)$	Error of $z_1(t)$	Error of $z_2(t)$
0	1.0000	10.0000	1.0000	10.0000	0.0000	0.0000
0.1	0.9057	9.0574	0.9057	9.0574	0.0000	0.0000
0.2	0.8204	8.2037	0.8204	8.2037	0.0000	0.0000
0.3	0.7430	7.4304	0.7430	7.4304	0.0000	0.0000
0.4	0.6730	6.7301	0.6730	6.7301	0.0000	0.0000
0.5	0.6096	6.0957	0.6096	6.0957	0.0000	0.0000
0.6	0.5521	5.5211	0.5521	5.5211	0.0000	0.0000
0.7	0.5001	5.0007	0.5001	5.0007	0.0000	0.0000
0.8	0.4529	4.5294	0.4529	4.5294	0.0000	0.0000
0.9	0.4102	4.1025	0.4102	4.1025	0.0000	0.0000
1	0.3716	3.7158	0.3716	3.7158	0.0000	0.0000

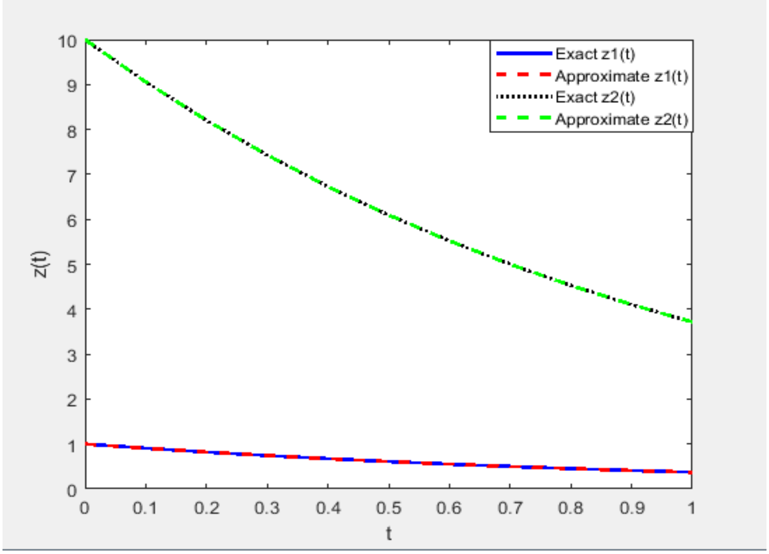

Figure (2)
Exact and Approximate solution of Example (2) (when N=8)

Conclusion

In this study, we successfully stratify the DTM to get numerical solutions for stiff system ordinary differential equations. It is watched that is an active and dependable instrument for the solution of stiff system ordinary differential equations. The method gives speedy converging chain solutions. The precision of the acquired solution can be improved by taking more expression in the solution. In many cases, the series solutions gained with DTM can be written in exact locked form. The offered method reduces the computational difficulties of the other conventional methods and all the computations can be made humble manipulations. The examples were examined by stratify the DTM and the consequence have a shown wonderful execution**.**

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